Journal of Pure and Applied Mathematics: Advances and Applications
Volume 9, Number 2, 2013, Pages 69-79

A NOTE ON SETS V_t^-, V_t^0, V_t^+ OF A SIMPLE GRAPH G WITH $\delta(G) \geq 2$

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Abstract

Let G be a simple graph with no isolated vertices. A subset D of a vertex set $V(G)$ is said to be a total dominating set of G if for every vertex $v \in V(G)$, there is a vertex $u \in D$ such that uv is an edge. The minimum cardinality of a total dominating set is called the total domination number of G and it is denoted by $\gamma_t(G)$. If $\delta(G) \geq 2$, then for every vertex $v \in V(G)$, $\gamma_t(G - v)$ is well defined. For a vertex $v \in V(G)$, $\gamma_t(G - v)$ is either equal to $\gamma_t(G)$ or less than $\gamma_t(G)$ or greater than $\gamma_t(G)$. We get a partition $V(G) = V_t^- \cup V_t^0 \cup V_t^+$, where

$$V_t^- = \{v \in V \,/\, \gamma_t(G - v) < \gamma_t(G)\},$$

$$V_t^0 = \{v \in V \,/\, \gamma_t(G - v) = \gamma_t(G)\}, \text{ and}$$

2010 Mathematics Subject Classification: 05C.

Keywords and phrases: total dominating set, total domination number.

Supported by D.S.T. Project (SR/S4/MS: 357/06).

Received September 26, 2012

$$V_t^+ = \{v \in V / \gamma_t(G - v) > \gamma_t(G)\}.$$

In this paper, we obtain a necessary and sufficient condition for a vertex to be in $V_t^+ (V_t^-)$. We prove that if $v \in V_t^+$, then the induced subgraph $< N[v] >$ is not complete and $N(v) \cap V_t^- = \phi$. If $V_t^+ \neq \phi$, then $|V_t^+| < |V_t^0|$. If $V(G) = V_t^-$ and $n = (\gamma_t(G) - 1)\Delta(G) + 1$, then we show that G is regular.

1. Introduction

We consider only simple finite undirected graphs without isolated vertices. For graph theoretic terminology, we refer to [2]. If x is a positive real number, $\lfloor x \rfloor$ and $\lceil x \rceil$ denote, respectively, the integral part of x, and the least integer not less than x. To each vertex v of a graph G, $N(v)$ denote the set of all vertices of G, which are adjacent to v, and $N[v] = N(u) \cup \{v\}$.

A set D of vertices of a graph G is said to be a *dominating set*, if every vertex in $V - D$ is adjacent to some vertex in D. We call D a total dominating set for G, if every vertex in V is adjacent to some vertex in D. The minimum cardinality of a dominating set (a total dominating set) of G is denoted by $\gamma(G)$, $(\gamma_t(G))$ and is called the *domination number* (the total domination number) of G. The concept of the total domination number was introduced by Cockayne et al. [3]. For further results, one can refer [1, 3, 4, 5, 6].

In this note, we study the effect of the removal of a vertex on the total domination number.

2. Definition and Main Results

Let G be a graph with $\delta(G) \geq 2$. Then $\gamma_t(G - v)$ is defined for all $v \in V(G)$. For a vertex $v \in V(G)$, either $\gamma_t(G - v) = \gamma_t(G)$ or $\gamma_t(G - v) < \gamma_t(G)$ or $\gamma_t(G - v) > \gamma_t(G)$. Let $V(G) = V_t^- \cup V_t^0 \cup V_t^+$, where

$$V_t^- = \{v \in V / \gamma_t(G - v) < \gamma_t(G)\},$$

$V_t^0 = \{v \in V / \gamma_t(G - v) = \gamma_t(G)\}$, and

$V_t^+ = \{v \in V / \gamma_t(G - v) > \gamma_t(G)\}$.

Example 2.1. For the graph G given in the Figure 1,

$$V_t^+ = \{1, 2, 3, 4\}; \quad V_t^- = \phi; \quad V_t^0 = \{5, 6, 7, 8, 9, 10\}.$$

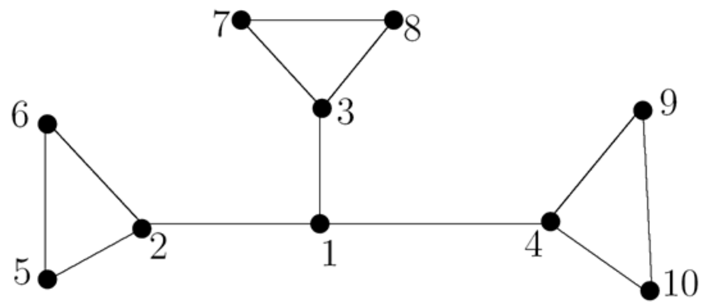


Figure 1

Example 2.2. For the graph G given in the Figure 2,

$V_t^+ = \{1, 2\}$; $V_t^- = \{4\}$; and $V_t^0 = \{3, 5, 6, 7\}$. (Note that $\gamma_t(G) = 3$.)

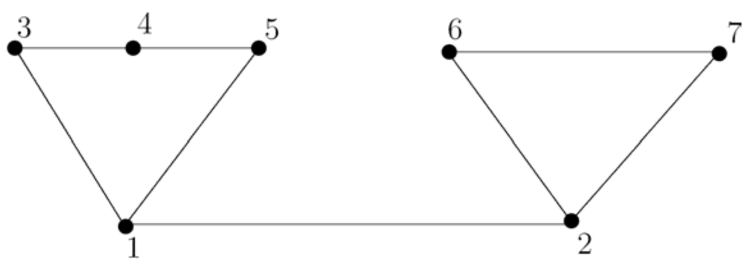


Figure 2

Example 2.3. For the cycle C_5, $V = V_t^-$ and for the cycle C_{4n}, $V = V_t^0$. If S is a γ_t-set of G, then the removal of any vertex in $V - S$ can not increase the total domination number and hence $V_t^+ \subseteq S$ for every γ_t-set S of G, and $|V_t^+| \leq \gamma_t(G)$. For the graph G given in Example 2.1, $|V_t^+| = \gamma_t(G)$. So, this upper bound for $|V_t^+|$ is attained.

Remark 2.4. V_t^+ need not be equal to $\bigcap\{S / S$ is a γ_t - set of $G\}$. In other words, if a vertex $v \in S$, for every γ_t - set S of G, then it is not necessary that $v \in V_t^+$. In the graph given in Figure 3, $v \in S$ for all γ_t - set S of G, yet $v \notin V_t^+$. (In fact, for this graph $V = V_t^0$; $V_t^+ = V_t^- = \phi$.)

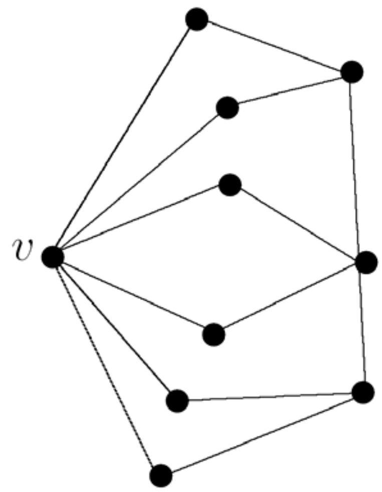


Figure 3

Now we prove the following theorem:

Theorem 2.5. *A vertex v of a graph G,* $(\delta(G) \geq 2)$, *is in* V_t^+ *iff*

(i) $v \in S$, *for every* γ_t *- set of G, and*

(ii) *if D is a subset of* $V - N[v]$ *with* $|D| \leq \gamma_t(G)$, *then D is not a* γ_t *- set for* $G - v$.

Proof. Let $v \in V_t^+$. Then by our earlier remark $v \in S$, for every γ_t - set of G. If $D \subseteq V - N[v]$ and D is a γ_t - set for $G - v$, then as $v \in V_t^+$, we have $|D| \geq 1 + \gamma_t(G)$, hence (ii).

Conversely, assume that conditions (i) and (ii) are satisfied. If $v \notin V_t^+$, then $v \in V_t^0 \cup V_t^-$. So, there exists a set $D \subseteq V - v$ with

$|D| \leq \gamma_t(G)$ and D is a γ_t - set for $G - v$. If $D \cap N[v] \neq \phi$, then D itself is a γ_t - set for G not containing v, a contradiction to (i). If $D \cap N[v] = \phi$, we get a contradiction to (ii). Thus $v \in V_t^+$. □

Theorem 2.6. *For any graph G with* $\delta(G) \geq 2$, *if* $v \in V_t^+$, *then*

(i) $PN_t(v, S) = \{u \in V(G) / N(u) \cap S = \{v\}\}$ *contains at least two vertices for any* γ_t *- set S of G.*

(ii) $\langle N[v] \rangle$ *is not complete.*

(iii) $N(v) \cap V_t^- = \phi$.

Proof. (i) Assume that for some γ_t - set S of G, $PN_t(v, S) = \{u\}$. As $\deg(u) \geq 2$ select a vertex $w \neq v \in N(u)$. Then $w \notin S$ and w is dominated by $S - v$, (as $w \notin PN_t(v, S)$). Now $(S - v) \cup \{w\}$ is a γ_t - set of G, not containing the vertex v, which is a contradiction to the fact that $u \in V_t^+$. So $|PN_t(v, S)| \geq 2$.

(ii) If $\langle N[v] \rangle$ is complete, consider any γ_t - set S of G. Clearly $v \in S$ and $|N[v] \cap S| \geq 2$. If $|N[v] \cap S| \geq 3$, then $S - v$ is a total dominating set for G, which is a contradiction. Hence $|N[v] \cap S| = 2$. As $2 \leq |PN_t(v, S)| \leq |N(v)|$, there is a vertex $w \in N(v) - S$. Then $(S - v) \cup \{w\}$ is a γ_t - set for G, which is a contradiction to $v \in V_t^+$. Thus $N[v]$ is not complete.

(iii) If possible let $u \in N(v) \cap V_t^-$. Let D be a γ_t - set of $G - u$. As $u \in V_t^-$, $|D| = \gamma_t(G) - 1$ and D is not a γ_t - set of G. So $D \cap N(u) = \phi$, in particular $v \notin D$. Select a vertex $w \neq v \in N(u)$, (note that $\deg(u) \geq 2$). As $N(w) \cap D \neq \phi$, the set $D \cup \{w\}$ is a γ_t - set for G, not containing the vertex v, which is a contradiction to $v \in V_t^+$. Thus, we have $N(v) \cap V_t^- = \phi$.

□

Remark 2.7. If $v \in V_t^+$, then $|PN_t(v, S)| \geq 2$, for every γ_t - set S of G. The induced subgraph $\langle PN_t(v, S) \rangle$ may be complete. For example, for the graph G given in the following Figure 4, $v \in V_t^+$ and $\langle PN_t(v, S) \rangle$ is complete for (any) γ_t - set S of G. Also note that $u \in V_t^+$ and $PN_t(u, S) \subseteq V_t^+$, where $S = \{u, v, w\}$ is the unique γ_t - set of G. In fact, $N[u] \subseteq V_t^+$.

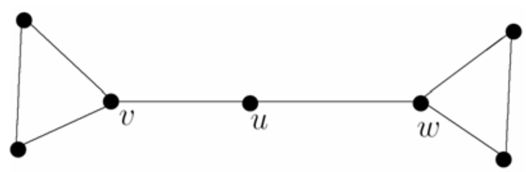


Figure 4

In Figure 3, we have given an example for a graph G with $\delta(G) \geq 2$ and $V_t^0 = \phi$. For these graphs, $V_t^+ = \phi$. In the following theorem, we prove that $V_t^+ \neq \phi$ implies $V_t^0 \neq \phi$.

Theorem 2.8. *For a graph G with $\delta(G) \geq 2$ and $V_t^+ \neq \phi$, $|V_t^+| < |V_t^0|$.*

Proof. It is enough to prove the theorem by assuming G is connected. For a γ_t - set S of G, define $A(S)$ and $B(S)$ as $A(S) = \{v \in V_t^+ / PN_t(v, S) \subseteq V_t^+\}$ and $B(S) = V_t^+ - A(S)$. Then $V_t^+ = A(S) \cup B(S)$. As $V_t^+ \neq \phi$, either $A(S) \neq \phi$ or $B(S) \neq \phi$.

Let $v \in A(S)$. As $PN_t(v, S) \subseteq V_t^+ \subseteq S$, $PN_t(v, S) \subseteq N(v) \cap S$ and as $|PN_t(v, S)| \geq 2$, we have $v \notin PN_t(u, S)$ for any $u \in PN_t(v, S)$. This shows that $PN_t(u, S) \cap S = \phi$; $PN_t(u, S) \subseteq V_t^0$ and $u \in B(S)$, for all $u \in PN_t(u, S)$. Thus, we proved that

(1) $A(S) \neq \phi$ implies $B(S) \neq \phi$, and

(2) for every $v \in A(S)$,

$$PN_t(v, S) \subseteq V_t^0 \text{ for all } u \in PN_t(v, S). \tag{1}$$

From (1), we have

$$V_t^+ \neq \phi \text{ implies } B(S) \neq \phi \text{ for all } \gamma_t\text{-set } S \text{ of } G. \tag{2}$$

Case (i). Assume that $A(S) \neq \phi$ for some γ_t-set S of G. Let $B(S) = \{v_1, v_2, \ldots, v_k\}$. To each $v_i \in B(S)$, select a vertex $u_i \in PN_t(v_i, S) \cap V_t^0$. To each $v \in A(S)$, let $A^0(v) = \cup\{PN_t(v_i, S) - \{u_i\} / v_i \in PN_t(v, S)\}$. Note that for every $v_i \in PN_t(v, S)$, $PN_t(v_i, S) \subseteq V_t^0$ and $|PN_t(v_i, S)| \geq 2$. So $|A^0(v)| \geq |PN_t(v, S)| \geq 2$, for all $v \in A(S)$. The sets $\{u_1, u_2, \ldots, u_k\}$, $A^0(v)$ for $v \in A(S)$ are disjoint subsets of V_t^0. So $|V_t^0| \geq |B(S)| + 2|A(S)| = |V_t^+| + |A(S)| > |V_t^+|$, as $A(S) \neq \phi$.

Case (ii). Assume that $A(S) = \phi$, for all γ_t-set S of G. Then $V_t^+ = B(S)$, for all γ_t-set S of G. Then for every $v \in V_t^+ (= B(S))$, $PN_t(v, S) \cap V_t^0 \neq \phi$ for any γ_t-set S of G. So if $|PN_t(v, S) \cap V_t^0| \geq 2$ for some $v \in V_t^+$ and for some γ_t-set S of G, then $|V_t^0| > |V_t^+| = |B(S)|$.

Now, further assume that $|PN_t(v, S) \cap V_t^0| = 1$ for all $v \in V_t^+$, and for all γ_t-set S of G. Fix any γ_t-set S of G. To each $v \in V_t^+$, select a vertex $v' \in V_t^+ \cap PN_t(v, S)$. As $N(v') \cap S = \{v\}$, we have $PN_t(v', S) \cap V_t^+ = \{v\}$ and hence $N(v) \cap S = \{v'\}$. Thus $\langle V_t^+ \rangle$ is a matching in G. Further, we note that if uv' is an edge in $\langle V_t^+ \rangle$, then

$$PN_t(v, S) = \{v'\} \text{ and } PN_t(v', S) = \{v\} \text{ for all } \gamma_t\text{-sets } S. \tag{3}$$

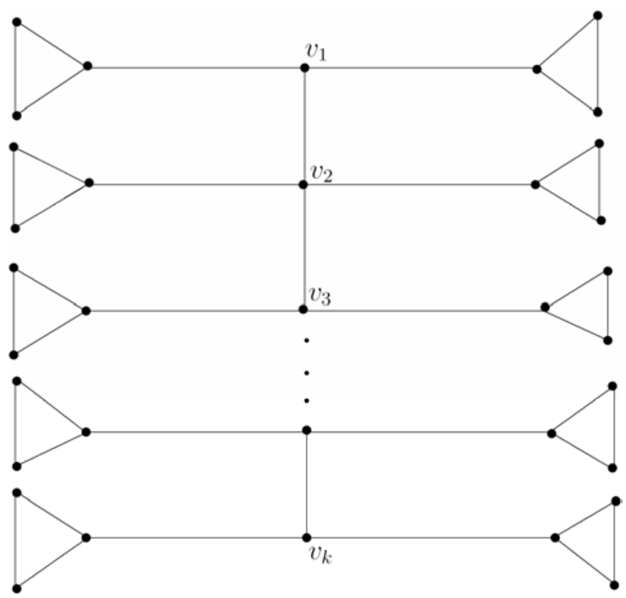
If $\deg(v) \geq 3$ for some $v \in V_t^+$, then $|N(v) \cap V_t^0| \geq 2$ and so $|V_t^0| \geq \sum_{u \in V_t^+} |N(u) \cap V_t^0| > |V_t^+|$. Now, we make an addition assumption that $\deg(v) = 2$, for all $v \in V_t^+$. Let $\{u\} = PN_t(v, S) \cap V_t^0$. (As $\deg(v) = 2$, $|PN_t(v, S) \cap V_t^0| = |PN_t(v, S) \cap V_t^+| = 1$, for all $v \in V_t^+$.) Let $C = \bigcup\{(PN_t(v, S) \cap V_t^0) / v \in V_t^+\}$. Then $|C| = |V_t^+| = |B|$. Note that C is invariant with respect to the choice of γ_t - set S, as we have assumed that $\deg(v) = 2$ in G for all $v \in V_t^+$. We claim that $V \neq B \cup C$. For if $B \cup C = V$, then the induced graph $\langle C \rangle$, contains no isolated vertex and hence C is a γ_t - set of G and C contains no element of V_t^+, which is a contradiction. Thus $V \neq B \cup C$ and hence, there is a vertex $w \notin B \cup C$ such that w is adjacent to some $u \in C$. As $w \notin B = V_t^+$, $w \in V_t^0 \cup V_t^-$. If $w \in V_t^-$, let D be a γ_t - set of $V - w$. Then $|D| = \gamma_t(G) - 1$ and $D \cap N(w) = \phi$. Hence $u \notin D$, but u is dominated by D. Therefore, $D \cup \{u\}$ is a total dominating set of G, with cardinality $\gamma_t(G)$. It follows that $D \cup \{u\}$ is a γ_t - set for G and if $N(u) \cap B = \{v\}$, there is a unique vertex $v' \in V_t^+$ adjacent to v (as $\langle V_t^+ \rangle$ is a matching and $B = V_t^+$). Now v, v', u belong to the γ_t - set $D \cup \{u\}$ of G, which is a contradiction to (3). Hence only possibility is $w \in V_t^0$. Thus, in this case, $|V_t^0| \geq |C \cup \{w\}| = |B| + 1 = |V_t^+| + 1$. Thus, we have proved that $|V_t^0| > |V_t^+|$ in all the cases.

□

Remark 2.9. (i) For the graph given in Figure 4, $|V_t^0| = 1 + |V_t^+|$.

(ii) If $|V_t^+| \neq \phi$ and $A = \{v \in V_t^+ / PN_t(v, S) \subseteq V_t^+\} \neq \phi$ for some γ_t - set S of G, by the Theorem 2.8, $|V_t^0| \geq |B| + 2|A|$, where $B = V_t^+ - A$.

The lower bound for $|V_t^0|$ is attained for the following graph given in Figure 5:



The graph G_k for which $|V_t^0| = 4k$.

Figure 5

Corollary 2.10. *For a graph* G *with* $\delta(G) \geq 2$, $V_t^0 = \phi$ *implies* $V = V_t^-$.

Remark 2.11. For a graph G with $\delta(G) \geq 2$, $V = V_t^- \Leftrightarrow G$ is a total *CVR*.

The following theorem characterizes the set V_t^-:

Theorem 2.12. *Let* G *be a graph with* $\delta(G) \geq 2$. *Then a vertex* $v \in V_t^-$ *iff there exists a* γ_t*-set* S, *such that* $v \notin S$ *and* $PN_t(u, S) = \{u\}$ *for some* $u \in S$.

Proof. Let $v \in V_t^-$. Then $\gamma_t(G - v) = \gamma_t(G) - 1$. Let D be a γ_t-set for $G - v$. Then $|D| = \gamma_t(G) - 1$ and hence D is not a total dominating set for G and $N(v) \cap D = \phi$. Select a vertex $w \in N(v)$. Then $D \cup \{w\}$ is a γ_t-set of G, such that $v \notin D \cup \{w\}$ and $PN_t(w, S) = \{v\}$, where $S = D \cup \{w\}$.

Conversely, assume that S is a γ_t-set of G, such that $v \notin S$ and $PN_t(u, S) = \{v\}$ for some $u \in S$. As $v \notin S$ and $PN_t(u, S) = \{v\}$, $S - u$ is a total dominating set for $G - v$. Hence $\gamma_t(G - v) = |S - u| = \gamma_t(G) - 1$. Therefore, $v \in V_t^-$. □

Theorem 2.13. *Let G be a simple graph with $\delta(G) \geq 2$. If $V(G) = V_t^-$ and $n = (\gamma_t(G) - 1)\Delta(G) + 1$, then G is regular.*

Proof. As $V = V_t^-$, for each vertex $v \in V$, $\gamma_t(G - v) = \gamma_t(G) - 1$. For each v, let S_v be a γ_t-set for $G - v$. Then

(i) $N(v) \cap S_v = \phi$.

(ii) $N(u) \cap S_v \neq \phi$ for all $u \in V - v$.

(iii) $2 \leq \deg(w) = |N(w) \cap (V - v)| \leq \Delta(G)$, for all $w \in S_v$.

As $|V - v| = \Delta(G)(\gamma_t(G) - 1) = \Delta(G)|S_v|$, it follows that $|N(u) \cap S_v| = 1$ for all $u \in V - v$, and $|N(w) \cap (V - v)| = \Delta(G)$ for all $w \in S_v$, and hence $\deg(w) = \Delta(G)$, whenever $w \in S_v$ for some $v \in V(G)$. Let $w \in S_v$. Then for each vertex $u \in S_v - w$, we have $|N(u) \cap S_w| = 1$ and hence $|N(S_v - w) \cap S_w| = |S_v - w| = |S_w| - 1$. So, there is exactly one vertex $x \in S_w$ such that $N(x) \cap (S_v - w) = \phi$. But as $x \in S_w$, $w \notin N(x)$, and hence $N(x) \cap S_v = \phi$. This is possible only when $x = v$, as S_v is a γ_t-set for $G - v$. Thus $w \in S_v \Rightarrow v \in S_w$ and hence if $u \in V$, then $u \in S_v$ for some v and $\deg(u) = \Delta(G)$ for all u, which is the required result. □

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